THE COPELAND-ERDŐS THEOREM ON NORMAL NUMBERS

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ABSTRACT. Let $b \geq 2$ be a positive integer. A real number α is called normal in base b if for every positive integer $k \geq 1$, every string of length k occurs in the decimal expansion of α in base b with frequency b^{-k}. Champernowne conjectured that the number

$$0.2357111317192329...,$$

whose decimal expansion in base 10 consists of all the primes in ascending order, is normal. This was proved in 1946 by Copeland and Erdős [4]. In this short note we present their original proof of this conjecture.

1. INTRODUCTION

Let $b \geq 2$ be a positive integer. A real number α is called normal in base b if for every positive integer $k \geq 1$, every string of length k occurs in the decimal expansion of α in base b with frequency b^{-k}. To make this definition precise, we need to specify what we mean by "frequency." Let us write $\alpha = \lfloor \alpha \rfloor + \{\alpha\}$, where $\lfloor \alpha \rfloor$ is the integer part of α and $\{\alpha\}$ is the fractional part of α. Suppose the decimal expansion of $\{\alpha\}$ in base b is given by

$$\{\alpha\} = 0.a_1 a_2 a_3 ... a_i a_{i+1} ...,$$

where $0 \leq a_i < b$ for all $i \geq 1$. For each digit $0 \leq c < b$, we define the frequency $f_c(\alpha, b)$ of c appearing in this decimal expansion by

$$f_c(\alpha, b) := \lim_{N \to \infty} \frac{\#\{i \leq N : a_i = c\}}{N},$$

provided this limit exists. We say that α is simply normal in base b if for every digit $0 \leq c < b$ we have $f_c(\alpha, b) = b^{-1}$. In other words, a number is simply normal if all possible digits occur equally often in its decimal expansion. It is easy to see that a number can be simply normal in one base but not in another. For example, the number $0.\dot{0}12345678\dot{9}$ is simply normal in base 10 but not in base 10^{10}. More generally, we can define the frequency $f_{c_1 c_2 ... c_k, b}(\alpha, b)$ of a string $c_1 c_2 ... c_k$ of length k appearing in the decimal expansion of α in base b by

$$f_{c_1 c_2 ... c_k}(\alpha, b) := \lim_{N \to \infty} \frac{\#\{i \leq N - k + 1 : a_i = c_1, a_{i+1} = c_2, ..., a_{i+k-1} = c_k\}}{N},$$

provided this limit exists. Regardless of the existence of this limit, we can always speak of the upper frequency $f^U_{c_1 c_2 ... c_k, b}(\alpha, b)$ and lower frequency $f^L_{c_1 c_2 ... c_k, b}(\alpha, b)$ of a string $c_1 c_2 ... c_k$ defined by

$$f^U_{c_1 c_2 ... c_k}(\alpha, b) := \overline{\lim_{N \to \infty}} \frac{\#\{i \leq N - k + 1 : a_i = c_1, a_{i+1} = c_2, ..., a_{i+k-1} = c_k\}}{N},$$

$$f^L_{c_1 c_2 ... c_k}(\alpha, b) := \underline{\lim}_{N \to \infty} \frac{\#\{i \leq N - k + 1 : a_i = c_1, a_{i+1} = c_2, ..., a_{i+k-1} = c_k\}}{N}.$$

Thus $f_{c_1c_2...c_k}(\alpha, b)$ exists if and only if $f^U_{c_1c_2...c_k}(\alpha, b) = f^L_{c_1c_2...c_k}(\alpha, b)$. We say that α is normal in base b if for every $k \geq 1$ we have $f_{c_1c_2...c_k}(\alpha, b) = b^{-k}$ for all possible strings $c_1c_2...c_k$ of length k. Statistically speaking, the digits of a normal number exhibit a complete randomness. Again, a number can be normal in one base but not in another. A real number is said to be absolutely normal if it is normal in any given base $b > 1$. It is easy to see that rational numbers are not normal in any base. Moreover, any infinite decimal in base $b \geq 3$ which lacks certain digit $0 \leq c < b$ cannot be normal. The set of such decimals is clearly uncountable. Consequently, the set of non-normal numbers in any given base $b \geq 3$ is uncountable.

It is not immediately clear whether normal numbers really exist. It is widely believed that the numbers $\sqrt{2}$ (or more generally, any irrational algebraic number), $e = 2.17828...$, and $\pi = 3.14159...$ are all normal in base 10, but no proof has been found. The existence of normal numbers was established by Borel [2] in 1909 who showed that for any $b > 1$ almost all real numbers are normal in base b. More precisely, he proved that the set of all real numbers which are not normal has Lebesgue measure 0. Hence from a measure theoretic point of view, not only do normal numbers exist but they take almost full share of the real line. For a modern proof of this result, see [6, §9.13]. A concrete example was discovered later by Champernowne [3] who showed that the number 0.1234567891011..., whose decimal expansion in base 10 consists of all positive integers in ascending order, is normal. This number is now known as Champernowne's constant. Just a year later, Besicovitch [1] proved that the same holds for the number 0.149162536496481..., whose decimal expansion in base 10 consists of all integral squares in ascending order. Champernowne conjectured that the number 0.2357111317192329..., whose decimal expansion in base 10 consists of all the primes in ascending order, is also normal. This was confirmed in 1946 by Copeland and Erdős [4]. The number 0.2357111317192329... is now referred to as the Copeland-Erdős constant. As a matter of fact, Copeland and Erdős proved the following stronger result.

Theorem 1.1. *Let $a_1 < a_2 < ...$ be a strictly increasing sequence of positive integers such that for every fixed real number $\theta \in (0, 1)$ we have*

$$A_N := \#\{i\colon a_i \leq N\} \gg N^\theta$$

for all sufficiently large N. Then the number

$$\alpha = 0.a_1a_2...a_na_{n+1}...$$

is normal in any base $b > 1$ when each a_i is expressed in its decimal expansion in base b.

For instance, we have by Chebyshev's estimate [6, Theorem 7] that

$$\pi(x) := \sum_{p \leq x} 1 \gg \frac{x}{\log x} \gg x^\theta$$

for every $\theta \in (0, 1)$, where the summation is over all primes $p \leq x$. It follows from Theorem 1.1 that the Copeland-Erdős constant 0.2357111317192329... is absolutely normal. More generally, let $q \geq 1$ be any positive integer and let $a \in \mathbb{Z}$ be an arbitrary integer coprime to q. Then we have analogously

$$\pi(x; q, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1 \gg \frac{x}{\log x} \gg x^\theta,$$

where the summation is over all primes $p \le x$ congruent to a modulo q. Thus Theorem 1.1 implies that the number $0.q_1q_2...q_nq_{n+1}...$, where $\{q_i\}_{i=1}^{\infty}$ is the sequence of all primes congruent to a modulo q arranged in ascending order, is absolutely normal. The same is true for the decimal consisting of all square-free positive integers in certain residue classes (mod q) arranged in ascending order. By Fermat's theorem on sum of two squares we have

$$\#\{n \le x \colon n = u^2 + v^2 \text{ for some } u, v \in \mathbb{Z}\} > \pi(x; 4, 1) \gg x^{\theta}$$

when x is sufficiently large. Hence Theorem 1.1 applies here as well.

At the end of their paper, Copeland and Erdős [4] conjectured that if $f \in \mathbb{R}[x]$ is a non-constant polynomial such that f maps positive integers to positive integers, then the number

$$0.f(1)f(2)...f(n)f(n+1)...$$

is normal in base 10. This was proved in 1952 by Davenport and Erdős [5] and hence includes the results of Champernowne and Besicovitch as special cases. A further generalization was provided by Nakai and Shiokawa [7] who showed that if $f(x)$ is a function of the form

$$f(x) = \alpha_0 x^{\beta_0} + \alpha_1 x^{\beta_1} + ... + \alpha_d x^{\beta_d},$$

where $\alpha_0, ..., \alpha_d \in \mathbb{R}$ and $\beta_0, ..., \beta_d \in \mathbb{R}$ with $\beta_0 > \beta_1 > .... > \beta_d \ge 0$, such that $f(x) > 0$ for all $x > 0$, then the number

$$0.\lfloor f(1) \rfloor \lfloor f(2) \rfloor ... \lfloor f(n) \rfloor \lfloor f(n+1) \rfloor ...$$

is absolutely normal. We will not dive into these results in the present note.

The remaining parts of this note are devoted to a presentation of the original proof of Theorem 1.1. We shall follow [4] with some adaptations. In the next section we shall introduce a couple of useful lemmas. We shall prove the theorem based on these results in the last section.

2. Preliminary Lemmas

In this section we shall prove three preliminary results. The proofs of Lemmas 2.1 and 2.2 below are due to the author himself and the proof of Lemma 2.3 follows [4].

Lemma 2.1. *A real number α is normal in base $b > 1$ if it is simply normal in base b^k for all positive integers k.*

Proof. We may assume $\alpha \in [0, 1)$. Let

$$\alpha = 0.a_1a_2a_3...a_ia_{i+1}...$$

be the decimal expansion of α in base b, where $0 \le a_i < b$ for all $i \ge 1$. Let k be any positive integer and put

$$\alpha_j = 0.a_ja_{j+1}a_{j+2}...$$

for $1 \le j \le k$. If $c_1c_2...c_k$ is any string of length k, where $0 \le c_1, ..., c_k < b$, and c is its decimal representation in base b^k, then c is a single digit. Note that

$$\begin{aligned} f_c(\alpha_j, b^k) &= \lim_{N\to\infty} \frac{\#\{i \le N \colon a_{(i-1)k+j} = c_1, a_{(i-1)k+j+1} = c_2, ..., a_{(i-1)k+j+k-1} = c_k\}}{N} \\ &= k \cdot \lim_{N\to\infty} \frac{\#\{i \le kN \colon i \equiv j \pmod{k}, a_i = c_1, a_{i+1} = c_2, ..., a_{i+k-1} = c_k\}}{kN} \end{aligned}$$

for all $1 \le j \le k$. It follows that

$$f_{c_1c_2...c_k}(\alpha, b) = \frac{1}{k}\sum_{j=1}^{k} f_c(\alpha_j, b^k), \tag{1}$$

provided that the frequencies in question exist. Since α is simply normal in base b^k, so is each α_j. Thus $f_c(\alpha_j, b^k) = b^{-k}$ for all $1 \le j \le k$. By (1) we have $f_{c_1c_2...c_k}(\alpha, b) = b^{-k}$. Hence α is normal in base b. □

Remark 1. The converse of Lemma 2.1 is also true. Namely, if α is normal in base $b > 1$, then it is simply normal in base b^k for all positive integers k. For a proof of this, see [8].

Let $\epsilon > 0$ be a positive real number and k a positive integer. A positive integer n, whose decimal expansion in base $b > 1$ is $n = a_1a_2...a_m$ with $a_1 \ne 0$, is called (ϵ, k)-normal in base b if $|f_{c_1c_2...c_k}(n, b, k) - b^{-k}| < \epsilon$ for all possible strings $c_1c_2...c_k$ of length k, where

$$f_{c_1c_2...c_k}(n, b, k) := \frac{\#\{i \le m-k+1 \colon a_i = c_1, a_{i+1} = c_2, ..., a_{i+k-1} = c_k\}}{m}.$$

This notion was first introduced by Besicovitch [1]. Suppose that $m \ge k$. Let c be the decimal expansion of $c_1c_2...c_k$ in base b^k. Put $n_j = a_ja_{j+1}...a_m$ for $1 \le j \le k$. Then

$$f_c(n_j, b^k, 1) = \frac{\#\{i \le m-k+1 \colon i \equiv j \pmod{k}, a_i = c_1, a_{i+1} = c_2, ..., a_{i+k-1} = c_k\}}{\lceil (m-j+1)/k \rceil},$$

where for any $x \in \mathbb{R}$, $\lceil x \rceil$ is the least integer greater than or equal to x. Thus we have

$$f_{c_1c_2...c_k}(n, b, k) = \frac{1}{m}\sum_{j=1}^{k} \left\lceil \frac{m-j+1}{k} \right\rceil f_c(n_j, b^k, 1). \tag{2}$$

It is obvious that (2) holds trivially when $m < k$. Since $\lceil x \rceil = \lfloor x \rfloor$ if $x \in \mathbb{Z}$ and $\lceil x \rceil = \lfloor x \rfloor + 1$ otherwise, we have

$$\sum_{j=1}^{k} \left\lceil \frac{m-j+1}{k} \right\rceil = k - 1 + \sum_{j=1}^{k} \left\lfloor \frac{m-j+1}{k} \right\rfloor.$$

Now we invoke the following identity of Hermite:

$$\sum_{i=0}^{q-1} \left\lfloor x + \frac{i}{q} \right\rfloor = \lfloor qx \rfloor \tag{3}$$

for any $x \in \mathbb{R}$ and positive integer q. We have

$$\sum_{j=1}^{k} \left\lfloor \frac{m-j+1}{k} \right\rfloor = \sum_{j=1}^{k} \left\lfloor \frac{m-k+1}{k} + \frac{k-j}{k} \right\rfloor = \sum_{j=0}^{k-1} \left\lfloor \frac{m-k+1}{k} + \frac{j}{k} \right\rfloor = m - k + 1.$$

It follows that

$$\frac{1}{m}\sum_{j=1}^{k} \left\lceil \frac{m-j+1}{k} \right\rceil = 1.$$

Combining this with (2) we obtain the following result.

Lemma 2.2. *Let $b > 1$ and $k \geq 1$ be positive integers and $\epsilon > 0$ a positive real number. Suppose that the decimal expansion of n in base b is $n = a_1 a_2 ... a_m$ with $a_1 \neq 0$. Put $n_j = a_j a_{j+1} ... a_m$ for $1 \leq j \leq k$. If n_j is $(\epsilon, 1)$-normal in base b^k for every $1 \leq j \leq k$, then n is (ϵ, k)-normal in base b.*

Remark 2. We give here a standard proof of Hermite's identity (3). Consider the function

$$f(x) := \sum_{i=0}^{q-1} \left\lfloor x + \frac{i}{q} \right\rfloor - \lfloor qx \rfloor$$

for $x \in \mathbb{R}$. It is easily seen that f is periodic of period $1/q$. Moreover, we have $\lfloor x + i/q \rfloor = \lfloor qx \rfloor = 0$ for all $x \in [0, 1/q)$ and all $0 \leq i < q$. Thus f vanishes on $[0, 1/q)$. We therefore conclude that f is identically 0, which proves (3).

By Lemma 2.2 we see that the number of positive integers $n \leq N$ which are not (ϵ, k)-normal in base b is at most

$$b^{k-1} \cdot \#\{n \leq N \colon n \text{ is not } (\epsilon, 1)\text{-normal in base } b^k\}. \tag{4}$$

We now prove the following result [4, Lemma] which provides a useful upper bound for the number of integers up to N which are not (ϵ, k)-normal in base b .

Lemma 2.3. *The number of positive integers up to N which are not (ϵ, k)-normal in base $b > 1$ is less than N^δ for all sufficiently large N, where $0 < \delta < 1$ depends only on b, ϵ, k.*

Proof. In view of (4), we need only to prove the lemma for $k = 1$ with $b \geq 2$ being arbitrary. Let us fix $\epsilon \in (0, 1/b)$. Let m be the unique positive integer for which $b^{m-1} \leq N < b^m$. Let E denote the set of positive integers $n \leq N$ whose decimal expansions in base b contain some digit $0 \leq c < b$ appearing fewer than $m(1-\epsilon)/b$ times or contain some digit $0 \leq c' < b$ appearing more than $m(1+\epsilon)/b$ times. Then

$$\#E \leq b \sum_{l < m(1-\epsilon)/b} \binom{m}{l} (b-1)^{m-l} + b \sum_{m(1+\epsilon)/b < l \leq m} \binom{m}{l} (b-1)^{m-l}.$$

For any positive integer $n \in [1, N] \setminus E$ and any $0 \leq c < b$, the number of times c appears in the decimal expansion of n in base b is between $m(1-\epsilon)/b$ and $m(1+\epsilon)/b$. Thus the number of total digits in the decimal expansion of n in base b is between $m(1-\epsilon)$ and $m(1+\epsilon)$. Since $\epsilon < 1/b \leq 1 - 1/b$, we obtain

$$-2\epsilon < \frac{-2\epsilon}{(1+\epsilon)b} \leq f_c(n, b, 1) - b^{-1} \leq \frac{2\epsilon}{(1-\epsilon)b} < 2\epsilon.$$

This shows that every $n \in [1, N] \setminus E$ is $(2\epsilon, 1)$-normal in base b.

It suffices to show that $\#E \leq N^\delta$ for sufficiently large N, where $\delta \in (0, 1)$ depends only on b and ϵ. To this end, let

$$h_l := \binom{m}{l} (b-1)^{m-l}$$

for $0 \leq l \leq m$. Then

$$\rho_l := \frac{h_l}{h_{l-1}} = \frac{m - l + 1}{(b-1)l} \tag{5}$$

for $1 \le l \le m$. It follows that h_l is increasing for $l \le (m+1)/b$ and decreasing for $l \ge (m+1)/b$. When N is sufficiently large, we have

$$\frac{m(1-\epsilon)}{b} < \left\lceil \frac{m(1-\epsilon/2)}{b} \right\rceil < \left\lfloor \frac{m+1}{b} \right\rfloor < \left\lceil \frac{m(1+\epsilon/2)}{b} \right\rceil < \frac{m(1+\epsilon)}{b} < m.$$

It follows that $\#E \le b(m+1)(h_{r_1} + h_{r_2})$, where $r_1 = \lfloor m(1-\epsilon)/b \rfloor$ and $r_2 = \lfloor m(1+\epsilon)/b \rfloor$. By (5) and the fact that ρ_l is strictly decreasing for $0 \le l \le m$, we have

$$b^m > h_{s_1} = h_{r_1} \prod_{l=r_1+1}^{s_1} \rho_l > h_{r_1} \rho_{s_1}^{s_1 - r_1} \ge h_{r_1} (\rho_{s_1})^{\frac{m\epsilon}{2b}},$$

where $s_1 = \lceil m(1-\epsilon/2)/b \rceil$. Similarly, we have

$$h_{r_2} = h_{s_2} \prod_{l=s_2+1}^{r_2} \rho_l < h_{s_2} \rho_{s_2+1}^{\frac{m\epsilon}{2b}} < b^m (\rho_{s_2})^{\frac{m\epsilon}{2b}},$$

where $s_2 = \lceil m(1+\epsilon/2)/b \rceil$. Note that $\rho_{s_1} > 1$ and $0 < \rho_{s_2} < 1$ for sufficiently large N. Moreover, we have

$$\rho_{s_1} - 1 = \frac{m - bs_1 + 1}{(b-1)s_1} \to \frac{b}{b-1}\left(\frac{1}{1-\epsilon/2} - 1\right) = \frac{b\epsilon}{(b-1)(2-\epsilon)} > 0$$

as $N \to \infty$. Similarly, we see that

$$1 - \rho_{s_2} \to \frac{b\epsilon}{(b-1)(2+\epsilon)} > 0$$

as $N \to \infty$. Thus there exists a constant $0 < C < b$ depending only on b and ϵ, such that $b\rho_{s_1}^{-\frac{\epsilon}{2b}} \le C$ and $b\rho_{s_2}^{\frac{\epsilon}{2b}} \le C$ for all sufficiently large N. It follows that

$$\#E < b(m+1)\left((b\rho_{s_1}^{-\frac{\epsilon}{2b}})^m + (b\rho_{s_2}^{\frac{\epsilon}{2b}})^m\right) \le 2C^m b(m+1) < b^{\delta(m-1)} \le N^\delta$$

for sufficiently large N, where $\delta \in (0,1)$ depends only on b and ϵ. This completes the proof of the lemma. □

3. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. In view of Lemma 2.1, it is sufficient to prove that α is simply normal in any base $b > 1$. Let $0 \le c < b$ and $\epsilon \in (0,1)$ be arbitrary. Let N be sufficiently large and m the unique positive integer for which $b^{m-1} \le N < b^m$. Since $\{a_i\}_{i=1}^\infty$ is strictly increasing and $A_N \ge C_\theta N^\theta$ for all $\theta \in (1-\epsilon, 1)$, where $C_\theta > 0$ is a constant depending on θ, there are at least $A_N - b^{m(1-\epsilon)} \ge C_\theta N^\theta - (bN)^{1-\epsilon}$ of a_i's up to N whose decimal expansions in base b have at least $m(1-\epsilon)$ digits. These numbers together have at least $m(1-\epsilon)(C_\theta N^\theta - (bN)^{1-\epsilon})$ digits in total. By Lemma 2.3, the number of a_i's up to N which are not $(\epsilon, 1)$-normal in base b is less than N^δ for some $0 < \delta < 1$ depending only on b and ϵ, provided N is sufficiently large. If $f_{c,N}(\alpha, b)$ denotes the frequency of c appearing in the decimal expansion of the string $a_1 a_2 ... a_{A_N}$ in base b, then

$$f_{c,N}(\alpha, b) < b^{-1} + \epsilon + \frac{mN^\delta}{m(1-\epsilon)(C_\theta N^\theta - (bN)^{1-\epsilon})}$$

for all sufficiently large N. Fix $\max(1-\epsilon,\delta)<\theta<1$. Then we have

$$f_c^L(\alpha,b)\le \varlimsup_{N\to\infty} f_{c,N}(\alpha,b)\le b^{-1}+\epsilon.$$

Since $\epsilon\in(0,1)$ is arbitrary, it follows that $f_c^L(\alpha,b)\le b^{-1}$, which holds for any positive integer $0\le c<b$. Note that

$$\sum_{c=0}^{b-1} f_c^L(\alpha,b)=1.$$

Hence we must have $f_c^L(\alpha,b)=b^{-1}$ for all $0\le c<b$. On the other hand, we have

$$\sum_{c=0}^{b-1} f_c^U(\alpha,b)=1$$

and $f_c^U(\alpha,b)\ge f_c^L(\alpha,b)=b^{-1}$ for all $0\le c<b$. It follows that $f_c^U(\alpha,b)=b^{-1}$ for all $0\le c<b$. Consequently, we have $f_c(\alpha,b)=b^{-1}$ for all $0\le c<b$. This shows that α is simply normal in base b.

Remark 3. From the proof above it is easily seen that Theorem 1.1 still holds if, instead of requiring the sequence $a_1,a_2,...$ to be strictly increasing, we only require it to be increasing with the additional property that for every fixed $\sigma>0$ we have

$$\#\{i\colon a_i=n\}\ll n^\sigma$$

for all positive integers n. Indeed, there are at least $C_\theta N^\theta-K_\sigma b^{1-\epsilon}N^{1-\epsilon+\sigma}$ of a_i's whose decimal expansions in base b have at least $m(1-\epsilon)$ digits, where $K_\sigma>0$ is a constant depending only on σ. Moreover, the number of a_i's up to N which are not $(\epsilon,1)$-normal in base b is less than $K_\sigma N^{\delta+\sigma}$. We may then fix $0<\sigma<\min(\epsilon,1-\delta)$ and proceed as before.

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